Algebra Exam. 2

Answer any 4 of the first 5 questions and the sixth question. Answer all questions with carefully reasoned and written proofs. This exam is for 3 hours. As general test taking strategy, answer questions that you find easier first and not in order. The first five questions are worth 10 marks each and the last is worth 20 marks.

I. Let G be a group and $\cong$ be an equivalence relation on G. We'll say that $\cong$ is **left-invariant** if for any $g, h, k \in G$, if $h \cong k$ then $gh \cong gk$. Prove:
(a) If $\cong$ is left-invariant and e is the identity element of G then $H = [e]$ is a subgroup of G.
(b) $a \cong b$ if and only if $b^{-1}a \in H$.
(c) Define **right-invariant** and show that H is a normal subgroup of G if and only if $\cong$ is also right invariant.

II. Let G be a group and H be a subgroup of G. Let X be the set of left-cosets of H in G, i.e., $X = \{gH : g \in G\}$. For any $a \in G$ define the map $\lambda_a : X \to X$ by $\lambda_a(gH) = agH$. Prove:
(a) Each λ_a is a 1-1, onto map of X to itself. Thus each λ_a is in the group $A(X)$ of all permutations of X.
(b) The map $\theta : G \to A(X)$ defined by $\theta(a) = \lambda_a$ is a group homomorphism.
(c) Identify necessary and sufficient conditions on H in order that θ be 1-1.

III. Let G be a cyclic group of order n. Prove that for every r that is a divisor of n, G has a unique subgroup of order r.

IV. Let G be the group $\mathbb{Z} \times \mathbb{Z}$, i.e., G is the set of all tuples (m, n) where $m, n \in Z$ and the operation on G is componentwise addition. Prove:
(a) If $\phi : G \to G$ is a homomorphism then there exist $a, b, c, d \in Z$ such that $\phi((m, n)) = (am + bn, cm + dn)$.
(b) ϕ is an automorphism if and only if $\begin{matrix} a & b \\ c & d \end{matrix}$ is in $GL_2(\mathbb{Z})$.
Hint: $(1, 0)$ and $(0, 1)$ generate G.

V. Consider the homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ defined by $\phi(n) = ([n], [n])$. Prove that it is onto if $gcd(p, q) = 1$. In this case identify $ker(\phi)$.

VI. For each of the following statements give either a proof or a counterexample:

(a) The groups of invertible upper triangular and invertible lower triangular matrices are isomorphic.

(b) If g is an element of infinite order in a group G and $\phi : G \to H$ is a homomorphism, then the order of $\phi(g)$ is either 1 or infinite.

(c) The group $SL_n(\mathbb{C})$ is a normal subgroup of the group $GL_n(\mathbb{C})$.

(d) If G is a finite group and $\phi : G \to H$ is a homomorphism, then upto isomorphism, there are only finitely many possibilities for the group $im(\phi)$.

(e) If H and K are subgroups of a group G, then so is HK.